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Fractions

Fractions are a rich part of mathematics, but we tend to manipulate fractions by rote rather than try to make sense of the concepts and procedures. Researchers have concluded that this complex topic causes more trouble for elementary and middle school students than any other area of mathematics (Bezuk and Bieck 1993). Teaching fractions is therefore both important and challenging. The National Council of Teachers of Mathematics recommends that instruction in fractions emphasize equivalent forms, estimating and comparing, and the reasonableness of results, not just correct answers and the steps in performing fraction algorithms. This chapter will help you make sense of a number of concepts related to fractions.

1. What Are Fractions?

Fractions were first used thousands of years ago, by both the Babylonians and the Egyptians, to answer questions involving how much and how many. Whereas at that time many quantities could be expressed using "counting" numbers (28 cattle, 176 scoops of rice), other situations, like sharing seven apples between two people or dividing an acre of land among four children, could not be described using counting numbers. A different type of number was needed to express amounts that were less than one but greater than zero: fractions. It wasn't until the Renaissance, however, that the use of fractions, as we know them today, became commonplace. More formally, fractions belong to the set of numbers known as the rational numbers—numbers that can be written in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$. (Because it isn't possible to divide by zero, division by zero is "undefined" in mathematics; see page 35 for an explanation.) In the set of rational numbers, a is called the *numerator* (from the Latin word meaning *number*) and b is called the *denominator* (from the Latin word meaning *namer*) of the fraction. The terms *fraction* and *rational number* are often used interchangeably. But the word *fraction* can be used to indicate a variety of things: a symbol of the form $\frac{a}{b}$, a notational system (*write that value as a fraction*), as well as a quantity that is not a rational number ($\frac{\pi}{10}$). Thus, not all numbers written using fraction notation actually represent rational numbers.

Interestingly, fractions have multiple meanings and interpretations. Educators generally agree that there are five main interpretations: fractions as parts of wholes or parts of sets; fractions as the result of dividing two numbers; fractions as the ratio of two quantities; fractions as operators; and fractions as measures (Behr, Harel, Post, and Lesh 1992; Kieren 1988; Lamon 1999).

- *Fractions as parts of wholes or parts of sets*. One meaning of fraction is as a part of a whole. In this interpretation, a unit is partitioned into equivalent pieces or a set is partitioned equally into smaller amounts (e.g., eighths, sixths, or halves) and numbers of these pieces are used to represent fractional amounts (e.g., three eighths, five sixths, one half). A pizza divided into equal-size pieces illustrates the part-of-a-whole meaning; a bushel of peaches separated equally into smaller boxes of peaches illustrates the parts-of-a-set meaning. In the parts-of-a-whole interpretation, the fractional parts do not have to be identical in shape and size (i.e., congruent), but they must be equivalent in some attribute such as area, volume, or number.
- *Fractions as the result of dividing two numbers*. A fraction can also represent the result obtained when two numbers are divided. This interpretation of fraction is sometimes referred to as the *quotient meaning*, since the quotient is the answer to a division problem. For example, the number of gumdrops each child receives when 40 gumdrops are shared among 5 children can be expressed as $\frac{40}{5}$, $\frac{8}{1}$, or 8; when two steaks are shared equally among three people, each person receives $\frac{2}{3}$ of a steak for dinner. We often express the quotient as a mixed number rather than an improper fraction—15 feet of rope can be divided to make two jump ropes, each $7\frac{1}{2}$ ($\frac{15}{2}$) feet long.
- *Fractions as the ratio of two quantities*. A ratio is a comparison between two quantities (see Chapter 8). When a ratio compares a part to a whole, the part-to-whole interpretation of fraction is being used. For example, if there are 15 children at a family gathering compared with a total of 33 people, we can write this comparison using a ratio (15:33), but we are more likely to refer to this part-to-whole relationship as a fraction ($\frac{15}{33}$). All fractions are ratios, but all ratios are not fractions. Why? Some ratios compare parts of a set to other parts of a set. For example, we can compare the 15 children with the 18 adults at the family gathering and then express the ratio of number of children to number of adults as 15:18, or 5:6. These part-to-part ratios are not fractions, because the ratio does not name a rational number but instead presents a comparison of two numbers. Furthermore, the formal definition of rational number indicates that zero is not allowed as the number in the denominator; the ratio 2:0 can be used to compare two blue marbles with zero green marbles, but it is not a fraction. Interestingly, students sometimes use part-to-part ratios to make sense of part-to-whole fractions.
- *Fractions as operators*. Here a fraction is understood to be a number that acts on another number in the sense of stretching or shrinking the magnitude of the number. A model plane, for example, can be $\frac{1}{25}$ the size of the original plane, or an image of a red blood cell might be magnified under a

microscope to 300 times its actual size. In these cases, a multiplicative relationship, or multiplication rule, exists between two quantities (e.g., if the length of the plane's wingspan is 50 feet and the length of the model's wingspan is $\frac{1}{25}$ times as long, then to find the wingspan of the model, we multiply $50 \times \frac{1}{25}$).

- *Fractions as measures*. The idea of a fraction as a length on the number line, created by partitioning units into subunits, is at the heart of this interpretation. A unit of measure can always be partitioned into smaller and smaller subunits. When we measure a distance using a ruler, we line up the object to be measured against hash marks. If the object doesn't line up precisely, however, this doesn't mean we can't measure its length! There is a dynamic aspect to the measurement interpretation of fraction—we name the fractional amount based on the number of subunits we are willing to create. Inherent to this interpretation of fraction is the understanding that there are an infinite number of rational numbers on the number line. We can always partition units and subunits into tinier and tinier subunits.

When a fraction is presented in symbolic form devoid of context, you cannot determine which interpretation of the fraction is intended. The various interpretations are needed, however, in order to make sense of fraction problems and situations. Furthermore, in many problem situations students are faced with results that must be interpreted using more than one meaning of fraction. For example, when two pizzas are shared among six children, the amount that each receives is $\frac{1}{3}$, which refers to the quotient meaning of fractions (the result of dividing two numbers), whereas the comparison of pizza to children (2 pizzas for 6 persons, or 1 for 3) is a ratio. If students are not familiar with meanings other than the part-whole interpretation, their understanding of situations like these is often incomplete.

When do students learn the different meanings of fractions? Research supports the idea that the part-whole interpretation, which involves partitioning wholes (or sets) into equal-size pieces and identifying different-size units, is the best way to approach learning about fractions in the early grades; students' experience with "fair shares" in everyday life is often the starting point for assisting students in understanding some important ideas about the size and number of units in a whole and how units can be divided up into smaller and smaller subunits. The other interpretations are best studied in more depth later in elementary and middle school, when connections between fractions and division and multiplication help students build on their initial understanding of part-whole relationships. However, this does not mean that students should not have experiences with all the other interpretations before fifth grade. On the contrary, students benefit from building a broad base of meaning and being able to move flexibly among meanings.

Fractions as Parts of Wholes or Parts of Sets

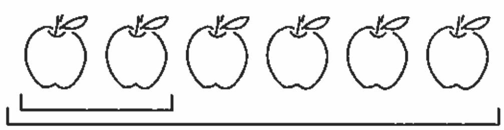
Since the interpretation of a fraction as a part of a whole is so fundamental to establishing a foundation for understanding fractions, let's examine three different relationships among parts and wholes.

One relationship involves being given the whole unit and the symbolic fraction and determining what the part looks like. We use this type of relationship every day. Suppose you want to put a portion of your earnings, say $\frac{1}{10}$, into a savings account. First you have to determine the "part" of the whole to save—in other words, what is $\frac{1}{10}$ of your weekly salary?

Let's take another example. If a whole set consists of twelve apples, how much is $\frac{1}{6}$ of the set? $\frac{3}{4}$ of the set? $\frac{5}{3}$ of the set?

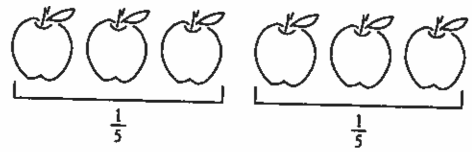
The fraction $\frac{1}{6}$ is called a unit fraction because it has a numerator of one. Unit fractions are the easiest for students to understand, one reason being that you only have to consider one part in relationship to the whole unit. Therefore, instruction in the early grades emphasizes unit fractions. In this case, two apples are $\frac{1}{6}$ of the set. Fractions that are less than one and have a numerator that is less than the denominator are called *proper fractions*. In this case, for the proper fraction $\frac{3}{4}$, you must divide the set into four equal subsets (three apples) and then combine three of the subsets, thereby ending up with nine apples. Fractions that are greater than one are called *improper fractions*. The improper fraction in this example, $\frac{5}{3}$, is equivalent to twenty apples (four apples are $\frac{1}{3}$, and $5 \times 4 = 20$).

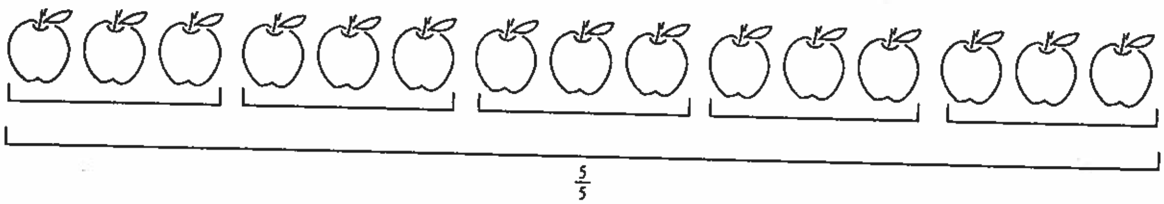
Another part-to-whole relationship involves being given the whole unit and the part and determining the fractional amount the part represents. For example, if six apples make up the whole set, what fraction of that whole is two apples?



To determine the fraction, you first have to partition the whole into equivalent parts. Two apples are equivalent to $\frac{1}{3}$ of a six-apple set, since the six apples can be divided into three equal groups of two apples each. Problems similar to this one, often using unit fractions only, can be used to introduce students to the concept of fractions. While this relationship is straightforward for adults, young children have difficulty partitioning sets into groups with more than one element in each (what fraction of six apples is five apples? $\frac{5}{6}$) and in understanding that $\frac{1}{3}$ and $\frac{2}{6}$ are equivalent. It is also important to explore relationships in which the part is greater than the whole: for example, the fractional relationship of ten apples to six apples is $\frac{10}{6}$, or $\frac{5}{3}$, or $1\frac{2}{3}$.

A third relationship in the part-to-whole interpretation involves determining the whole after being given the fraction and the part. This relationship can be especially difficult for students to grasp, because the size of the whole must be extrapolated from the part. For example, if $\frac{2}{5}$ of a whole is six apples, then how much is the whole set? Using the given information, it can be determined that $\frac{1}{5}$ is equivalent to three apples, and since there are five fifths in a whole set, multiplying five by three tells us how much. Thus, six is $\frac{2}{5}$ of fifteen. The relationship can be illustrated like this:





This type of part-to-whole relationship is common in many word problems. For example: *Eight puppies in a litter are all black. The remaining $\frac{1}{3}$ of the litter are spotted. How many puppies are in the litter?* (Answer: 12.) Many adults have difficulty with this type of problem because their understanding of fraction and fraction relationships is not robust. In order to help students develop numerical power with fractions, we need to include mathematical tasks in our instruction that require students to explore and make sense of these various relationships.

The examples above use the concept of sets to explore part-whole relationships. These part-whole relationships also can be investigated using regions divided into equal-size areas (such as rectangles or circles), or using linear models in which the lengths are divided, such as on a number line.

Activity

Exploring the Part-Whole Meaning of Fractions

Objective: investigate the relationships among part-whole fractions and the wholes using lengths as the wholes.

Materials: Cuisenaire rods.

Use Cuisenaire rods to answer these part-whole questions:

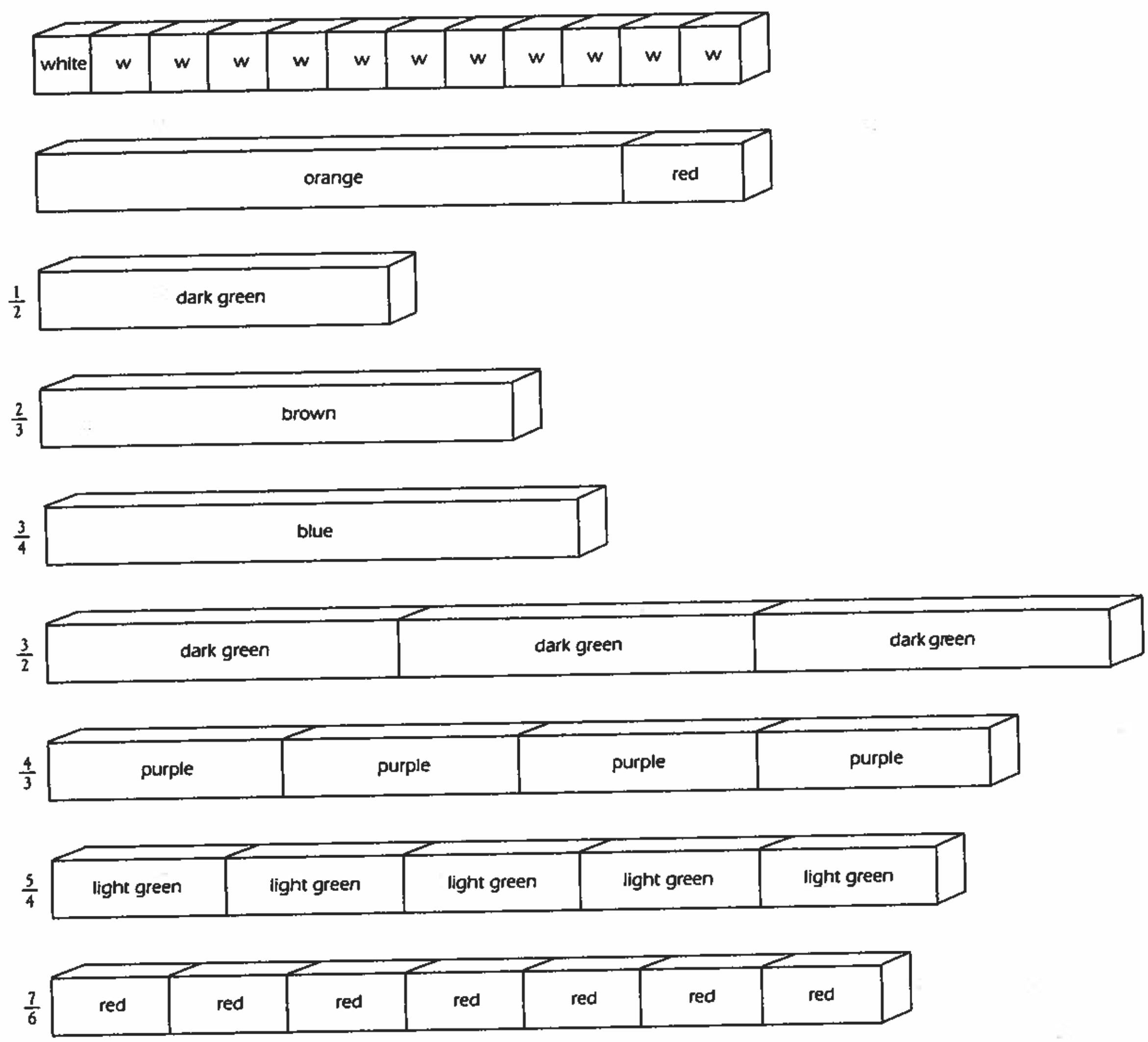
1. If we assign the blue rod a value of 1, what is the value of the light green rod?
2. If we assign the value of 1 to the light green rod, what is the value of the blue rod?
3. If the orange rod plus the red rod is one whole, which rods represent $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$?
4. If the orange rod plus the red is one whole, which rods represent $\frac{3}{2}$, $\frac{4}{3}$, $\frac{5}{4}$, and $\frac{7}{6}$?
5. Which fraction in Question 4 is the greatest? Is there a pattern? Which is greater, $\frac{26}{25}$ or $\frac{17}{16}$? Explain how you determined the greater fractions.
6. If the red rod is $\frac{1}{3}$, what rod is the whole?
7. If the dark green rod is $\frac{2}{3}$, what rod is the whole?
8. If the yellow rod is $\frac{5}{4}$, what rod is one whole?

Things to Think About

When manipulatives are used to explore part-whole relationships, teachers can directly observe what students understand about the relationships. Watching students solve these problems can be quite revealing. For example, in solving Question 1 do students represent the blue rod using three light green rods and indicate that each light green rod is $\frac{1}{3}$ of the blue rod, or do they use a variety of rods in addition to or instead of green ones to make a rod the same length as the blue one?

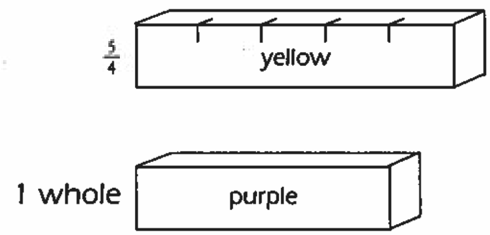
This activity raises a number of questions. Can a whole number be expressed as a fractional relationship? Yes. Whole numbers can be represented as fractions with denominators of one (e.g., 4 is equivalent to $\frac{4}{1}$). Thus, since the light green rod in Question 2 has been designated 1, the blue rod is 3, or $\frac{3}{1}$. Did using different rods to represent one whole confuse you? One difficulty children encounter when studying fractions is that different amounts can be used as the whole, and therefore the same fractions may not be identical in size. For example, half of a medium-size pizza is smaller than half of a large pizza, though both can be represented numerically as $\frac{1}{2}$ and the relationship represented by $\frac{1}{2}$ is the same in each case. It is important for children to pay attention to the size of the "whole" when they are dealing with fractions in context. Likewise, in order to compare two fractions, the referent whole must be identical for both.

The length of the orange rod plus the red one is equivalent to twelve white rods. These rods can be used to demonstrate many fractional relationships. The various fractional relationships in Questions 3 and 4 are shown pictorially below.



Notice that of the four fractions in Question 4, $\frac{3}{2}$ is the greatest. Each of these improper fractions is just one fractional part greater than one (e.g., $\frac{7}{6}$ is equivalent to $1\frac{1}{6}$: it is one sixth larger than one). Thus, $\frac{3}{2} > \frac{4}{3} > \frac{5}{4}$, since in terms of the extra fractional piece greater than one, $\frac{1}{2} > \frac{1}{3} > \frac{1}{4}$. Using this line of reasoning, $\frac{17}{16} > \frac{26}{25}$.

The last three questions require you to determine the whole or unit given the part and its fractional value. If the red rod is $\frac{1}{3}$, then the dark green rod is the whole, because three reds are the same length as a dark green. If the dark green rod is $\frac{2}{3}$, then the blue rod is the whole. The answer to Question 8 can be explained using a diagram:



Since the yellow rod is five fourths of one, we need to find a rod equivalent to four fourths. It is the purple rod. ▲

In order for students to attach meaning to fractions and make sense of the various part-whole relationships, a variety of physical materials should be introduced and explored during instruction. The pictorial representations included in most textbooks are not sufficient. Consider how difficult it would have been to answer some of the questions in Activity 1 if you only had drawings of the rods. When investigating fractions as a part of a whole, the fractional parts can be represented using cardboard or paper circles cut into wedges, cardboard or paper rectangles cut into smaller rectangles, drawings on dot or graph paper, areas on geoboards, sections in a piece of folded paper, Cuisenaire rods, pattern blocks, distances on a ruler or number line, and other commercial materials. Working with a variety of discrete objects—counters, beans, marbles, Xs and Os—can help students see parts of a set. One important feature of fraction instruction is that mathematical tasks be meaning oriented rather than symbol oriented.

Fractions as the Result of Dividing Two Numbers

A fraction can be viewed as the result of a division. For example, $\frac{1}{3}$ can be interpreted as 1 whole unit or item (candy bar, pizza, money) divided into 3 equal parts. Partitioning, the process of dividing an object or set into parts, is the essential activity that underlies students' ability to make sense of this interpretation of fraction. Children progress through stages in partitioning; first, they become competent at cutting items in half. They then move to dividing units into fourths, eighths, and sixteenths by halving successively. Later students figure out the more complex process of how to divide items into an odd number of pieces. Interpreting a fraction as the result of division, or as a quotient, occurs in the elementary curriculum when we ask students to share small quantities with many others (2 cookies shared among 4 people) and when students represent quotients with fractions instead of whole-number remainders. For example, when dividing 7 by 5, we can represent the answer as $\frac{7}{5}$ or $1\frac{2}{5}$. When dividing 1 by 5, we can represent the answer as $\frac{1}{5}$.