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Multiplication and Division

Multiplication and division have always been important topics in elementary school mathematics. However, in the past, instruction has focused primarily on helping students develop procedural competency with basic facts and paper-and-pencil algorithms. This procedural competency is an important goal (and suggestions for helping students become competent with multiplication and division facts and algorithms are included in Chapter 2), but recent research makes it clear that students also need to develop deep conceptual knowledge of multiplication and division in order to apply and use these operations to solve problems. The National Council of Teachers of Mathematics recommends that students in grades 3 through 5 should “develop a stronger understanding of the various meanings of multiplication and division, encounter a wide range of representations and problem situations that embody them, learn about the properties of these operations, and gradually develop fluency in solving multiplication and division problems” (NCTM 2000, 149). In middle school, students solve a wide range of problems using these operations and learn to multiply and divide signed numbers in preparation for algebra.

1. Types of Multiplication and Division Problems

Think back to the childhood instruction you received in multiplication and division. The focus may have been on performing multidigit calculations accurately. Students’ ability to “do” multiplication and division calculations—to apply step-by-step procedures that result in correct solutions—implies that they have procedural knowledge of these operations. However, simply being able to perform calculations does not necessarily mean that students understand these operations. Conceptual knowledge is based on understanding relationships—in this case, relationships that represent multiplication and division. These relationships can be expressed using pictures, graphs, objects, symbols, and words. For example, when students encounter a word problem and are asked to translate the relationship into symbols, their conceptual knowledge of the situation helps them do so. Since everyday mathematics is almost always applied in the context of words, not symbols, it is important for students to understand the relationships inherent in multiplication and division problems. In addition, students’ understanding of these relationships helps them generalize their knowledge and apply it to related concepts in algebra.

Likewise, the language of multiplication and division situations must be understood. In a multiplication equation such as $3 \times 2 = 6$, the 3 and the 2 are *factors*. The 6, which is the solution to this equation, is the *product*. In division we sometimes convert the operation to a multiplication equation ($2 \times \square = 6$, if we are dividing 6 by 2) and refer to the unknown as the missing factor, but more often we use separate labels. In the equation $6 \div 2 = 3$, for example, the 6 is the *dividend*, the 2 is the *divisor*, and the 3 (the answer) is the *quotient*. In some division expressions ($7 \div 2$, for example), there is also a remainder, which can be expressed in a number of different ways: as a whole number, a decimal, or a fraction. In the example given, the quotient can be represented as 3 remainder 1, 3.5, or $3\frac{1}{2}$. The relationship between these terms can be expressed generically this way: *factor* \times *factor* = *product* and *dividend* \div *divisor* = *quotient* + *remainder*. Middle school students who have difficulty rewriting division sentences with and without remainders as multiplication sentences may not be secure in their understanding of the inverse relationship between the two operations. For example, the sentence $7 \div 2 = 3 \text{ r } 1$ is related to $(2 \times 3) + 1 = 7$, and $7 \div 2 = 3\frac{1}{2}$ is related to $2 \times 3\frac{1}{2} = 7$.

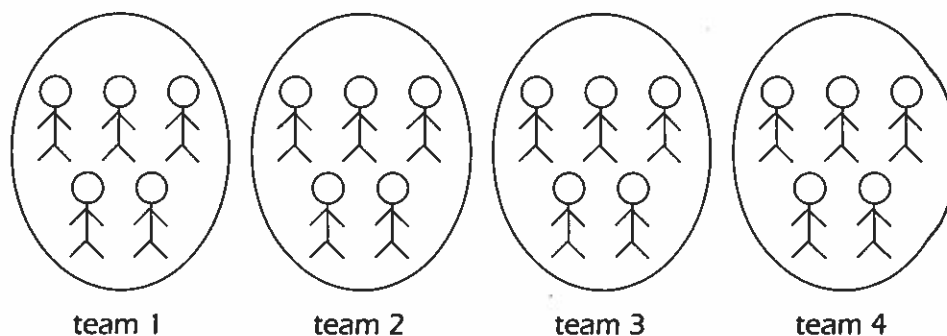
Researchers have defined and classified multiplication and division problems in a number of different ways based on their semantic structure—that is, how the relationships are expressed in words. The semantic structure of problems differs with regard to the nature of the quantities used and the quantity that serves as the unknown. The semantic structure of multiplication and division problems has two broad categories: asymmetrical and symmetrical problems. In asymmetrical situations, the quantities play different roles. For example, in the problem *There are 15 cars in the parking lot and each car has 4 tires. How many tires are there in all?*, 4 represents the amount in one group, 15 represents the number of groups and also acts as the multiplier. These roles are not interchangeable. If you switch the numbers (4 cars with 15 tires each) you have a different problem: the number of things in one group is 15 and the number of groups is now 4. (In an asymmetrical problem the answer is the same regardless of the role of the quantities, but that is not obvious to children.) In symmetrical situations, on the other hand, the quantities have interchangeable roles. In symmetrical multiplication problems it is not clear which factor is the multiplier. For example, in the problem *What is the area of a room that is 10 feet by 12 feet?*, either number can be the width or the length and either can be used as the multiplier.

There are three subcategories of asymmetrical problems: (1) equal grouping, (2) rate, and (3) multiplicative compare. There are two subcategories of symmetrical problems: (1) rectangular array and (2) Cartesian product. Each subcategory includes both multiplication and division problems, depending on which quantity is unknown. Students need to work with all types of multiplication and division problems in order to make sense of the relationships inherent to each type and to extend their understanding of these operations beyond mere procedures. As a teacher you need to guide students in discerning the role of the quantities within problems.

Equal Grouping Problems

Usually when we think of the operation of multiplication, an equal grouping problem comes to mind. In equal grouping multiplication problems, one factor tells the number of things in a group and the other factor tells the number of equal-size

groups. This second factor acts as a multiplier. For example, in the problem *There are four basketball teams at the tournament and each team has five players. How many players are at the tournament?*, the factor 5 indicates the number of players in one group and the factor 4 indicates the number of equal groups of 5. In this case, the 4 acts as the multiplier. Equal grouping problems are easy to model with pictures or by using repeated addition:



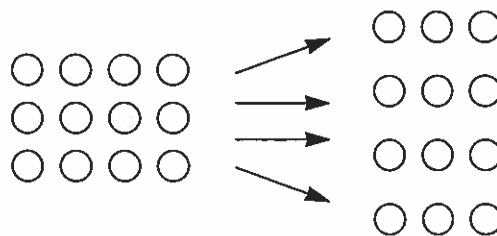
Two situations result in a problem's being classified as an equal grouping division problem—either the number of groups is unknown or the number in each group is unknown. These two types of division situations are referred to as *quotitive division* and *partitive division*, respectively.

Here is a partitive division problem:

Twenty-four apples need to be placed into eight paper sacks. How many apples will you put in each sack if you want the same number in each sack?

The action involved in partitive division problems is one of dividing or partitioning a set into a predetermined number of groups. If students model this situation, 24 objects are evenly distributed into 8 different paper sacks or groups.

When teaching division, teachers often choose partitive division examples to highlight equal sharing. For example, students are instructed to divide a set number of counters into four equal groups by distributing the counters one at a time into four piles:



Yet if partitive division problems are used exclusively in instruction, students often have difficulty making sense of quotitive division problems. Their mental model of what division is all about does not include this other meaning.

In quotitive division problems (sometimes referred to as *repeated subtraction* problems) the number of objects in each group is known, but the number of groups is unknown. For example: *I have 24 apples. How many paper sacks will I be able to fill if I put*

3 apples into each sack? The action involved in quotitive division is one of subtracting out predetermined amounts. If asked to model this problem, students usually repeatedly subtract 3 objects from a group of 24 objects and then count the number of groups of 3 they removed (i.e., 8). (In partitive division, on the other hand, they “divide” the 24 objects into 3 groups.)

The standard long division algorithm uses the quotitive interpretation of division: the divisor represents the number in one group, and this amount is repeatedly subtracted from the dividend. The number of multiples (or groups) of the divisor that are subtracted from the dividend is the answer. Students benefit from exposure to both types of division examples so that they internalize that two actions, subtracting and partitioning, are used to find quotients.

Rate Problems

Rate problems involve a rate—a special type of ratio in which two different quantities or things are compared. Common rates are miles per gallon, wages per hour, and points per game. Rates are frequently expressed as unit rates—that is, one of the quantities in the ratio is given as a unit (e.g., price per single pound or miles per single hour). In rate problems, one number identifies the unit rate and the other tells the number of sets and acts as the multiplier. For instance, in the problem *Concert tickets cost seven dollars each. How much will it cost for a family of four to attend the concert?*, the unit rate is seven dollars per single ticket (seven to one) and the multiplier is four. Rate problems can also be expressed as division situations. Here is a partitive division rate problem (size of one group is unknown): *On the Hollingers’ trip to New York City, they drove 400 miles and used 12 gallons of gasoline. How many miles per gallon did they average?* Quotitive division problems (number of equal groups is unknown) are also common in this category: *Jasmine spent \$108 on some new CDs. Each CD cost \$18. How many did she buy?*

Many students have had little experience with rates other than prices. As a result, they often are unsure of how to approach, interpret, and solve these types of problems. One way to help students understand rate multiplication problems is to have them calculate a unit rate (e.g., the number of feet they can walk or jog in one minute, the number of words they can read or write in one minute) and then apply this unit rate to various unit groupings (if I can read 45 words in one minute, I can read 90 words in two minutes, 135 words in three minutes, . . .). Through experiences, students learn which rates are extendable (the proportional relationship can be applied to all cases: if one bag of sugar costs \$2.59, we can predict the cost of 5 or 12 or 80 bags) and which rates are limited (the proportional relationship does not extend beyond a certain domain: if you can run 1 mile in 4 minutes, how long will it take to run a marathon [26 miles]?). Especially difficult for young students are rates involving speed and distance. Not only are the ideas somewhat hard to grasp, but we refer to these two concepts in a number of different ways—by using a rate (miles per hour or miles per gallon) or by using words and phrases such as *speed*, *distance*, *how fast*, and *how far*.

Multiplicative Compare Problems

The third type of asymmetrical problem is multiplicative compare, also called a *scalar problem*. Here, one number identifies the quantity in one group or set while the other

number is the comparison factor. For example, in the problem *Catherine read twelve books. Elizabeth read four times as many. How many books did Elizabeth read?*, the number 12 tells us the amount in a group and the number 4 tells us how many of these groups are needed. In multiplicative compare division problems, either the amount in each group or the comparison factor is missing. The following problems are multiplicative compare division problems:

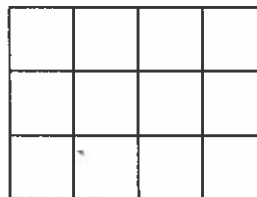
Elizabeth read 48 books during summer vacation. This is four times as many as Catherine. How many books did Catherine read during summer vacation?

Elizabeth read 48 books during summer vacation. Catherine read 12 books during summer vacation. How many times greater is the number of books Elizabeth read compared with the number of books Catherine read?

The relational language in multiplicative compare problems (e.g., *times as many*, *times greater*) is difficult for all students, especially so for those for whom English is a second language. Students make sense of both the language and the relationships the language implies by discussing and modeling these problems. Comparing and contrasting the language of additive relationships (*more than*, *less than*) with that of multiplicative relationships (*times more than*, *times as many*) may also be helpful to students. Despite the complex nature of these problems, the consensus among researchers is that early experience with multiplication and division should involve problems of this type.

Rectangular Array Problems

The rectangular array problem, a subcategory of symmetrical problem commonly known as an area problem, is often used to introduce the idea of multiplication. Students are presented with an array (e.g., three by four) and asked to label the two sides of the array and determine the total number of square units in the array:



In rectangular array problems the role of the factors is interchangeable. For instance, when finding the area of the array above, neither the three nor the four is clearly the multiplier.

Cartesian Product Problems

The other subcategory of symmetrical problems, known as Cartesian product problems, involves two sets and the pairing of elements between the sets. These problems entail a number of combinations. For example: *Pete's Deli stocks four types of cold cuts and two types of cheese. How many different sandwiches consisting of one type of meat and one type of cheese are possible?* In these problems, like the rectangular array problems, neither of the two factors is clearly the multiplier.