

# 3

## Addition and Subtraction

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In order to use addition and subtraction effectively, children must first attach meaning to these operations. One way for young children to do this is by manipulating concrete objects and connecting their actions to symbols. However, this is not the only way. They extend their understanding of situations involving addition and subtraction by solving word problems.

In this chapter we investigate how children solve addition and subtraction word problems involving small quantities and address several questions: How do children decide which operation is called for? (Is it an addition or subtraction situation?) How do children represent the mathematics symbolically? (Can they write an appropriate number sentence or equation to represent the situation?) How do children think numerically to perform the needed computation?

As students progress through our educational system, they are introduced to the set of integers and to addition and subtraction of signed numbers. In order to make sense of these operations, students must start to work with formal mathematical systems and expand their definitions of addition and subtraction to include directions of movements on the number line.

### 1. Types of Addition and Subtraction Word Problems

Students extend their understanding of and skill with addition and subtraction of whole numbers by solving word problems based on different meanings or interpretations of these operations. For example, consider these two subtraction problems:

*I have seven apples and Carla has four apples. How many more apples do I have than Carla?*

*I had seven apples and ate four apples. How many apples are left?*

Both situations can be expressed with the same number sentence:  $7 - 4 = \square$ . However, the first problem requires a “comparison” interpretation, the second, a “take away” interpretation. When an operation is reduced to symbols, it is impossible to determine which meaning or interpretation is being represented. So that students can learn that there are multiple interpretations of an operation and expand their

repertoire of situations that model the operation, we need to give them a variety of problems to solve.

Evidence suggests that the general meaning of a problem rather than specific words or phrases determines both the difficulty of the problem and the processes students use to solve it. In other words, it is how the operation is expressed and, by extension, a student's ability to make sense of that meaning rather than grammatical considerations such as the sequence of information and the presence of cue words that make problems easier or harder. In general, difficulties with word problems do not occur because students cannot read the words but because they cannot make sense of the mathematical relationships expressed by these words. Students' understanding of the different kinds of relationships in word problems is improved by solving and discussing problems.

A common classification scheme identifies four broad categories of addition and subtraction based on the type of action or relationship in the problems: join, separate, part-part-whole, and compare (Carpenter, Fennema, and Franke 1994; Carpenter, Fennema, Franke, Levi, and Empson 1999). Within these four broad categories, there are a total of eleven problem types. (Many of these problems use the same key words, even though their structure is different.) Often, textbooks include only one or two types of addition and subtraction problems. However, to become proficient and competent users of mathematics, students need to be able to solve all types of problems.

### *Join Problems and Separate Problems*

Join problems and separate problems involve actions that increase or decrease a quantity, respectively. In both categories, the change occurs over time. There is an initial quantity that is changed either by adding something to it or by removing something from it, resulting in a larger or smaller final quantity. There are three subsets of each of these problem types, depending on which quantity the solver is being asked to determine: the result, the amount of change, or the initial quantity.

| JOIN                     | EXAMPLE PROBLEM   |
|--------------------------|---|
| Result Unknown           | Laina had four dolls. She bought two more. How many dolls does she have now?<br>$4 + 2 = \square$   |
| Change Unknown           | Laina had four dolls. She bought some more dolls. Now she has six dolls. How many dolls did Laina buy?<br>$4 + \square = 6$                             |
| Initial Quantity Unknown | Laina had some dolls. She bought two more dolls. Now she has six dolls. How many dolls did Laina have before she bought some more?<br>$\square + 2 = 6$ |

| SEPARATE                 | EXAMPLE PROBLEM  |
|--------------------------|--|
| Result Unknown           | Rodney had ten cookies. He ate three cookies. How many cookies does Rodney have left?<br>$10 - 3 = \square$  |
| Change Unknown           | Rodney had ten cookies. He ate some of the cookies. Now he has seven cookies left. How many cookies did Rodney eat?<br>$10 - \square = 7$          |
| Initial Quantity Unknown | Rodney had some cookies. He ate three cookies. Now he has seven cookies left. How many cookies did Rodney have to start with?<br>$\square - 3 = 7$ |

In join problems and separate problems, the action of adding or subtracting is explicit. You can help students solve both types of problems by asking them to model these actions with objects. For example, using the join/amount-of-change-unknown problem above (*Laina had four dolls. She bought some more dolls. Now she has six dolls. How many dolls did Laina buy?*), students first can represent the four dolls using four blocks. Mimicking the action in the problem, students add some more blocks (dolls) until there are a total of six blocks. By then asking your students to explain how they determined the number of blocks to add, you help them form mental models of this type of situation as well. After solving many problems of this type, students eventually no longer need the physical model and can deal with the relationships symbolically.

Which quantity in a problem is unknown contributes to the overall difficulty of the problem. In general, when the result is unknown, students are more likely to be able to make sense of the relationships. Students are most familiar with result-unknown problems, because they encounter many similar problems in their everyday lives. These types of problems are also more heavily represented in textbooks.

Join problems and separate problems in which the initial quantity or the amount of change is unknown are more difficult for students. One reason is that these problems are often presented in language that suggests one action (e.g., separating) but require using the opposite action (e.g., joining) to find the answer. For example, take the following separate problem in which the initial quantity is unknown: *Rodney had some cookies. He ate three cookies. Now he has seven cookies left. How many cookies did Rodney have to start with?* There is a separating action in the problem, but a child can solve it using addition, or by counting on from seven: "Eight, nine, ten. Rodney started with ten cookies." Similarly, some join problems are solved by subtracting. It's important to notice whether a child is able to recognize the operation that matches the situation, can represent the number sentence or equation correctly, and then can think numerically to find the answer. It is useful for teachers to talk with students about each aspect of problem solving: *What is the problem describing? How can you write that down? How can you find the answer?*

### Part-Part-Whole Problems

Part-part-whole problems do not use action verbs—action neither occurs nor is implied. Instead, relationships between a particular whole and its two separate parts are established. There are two types of part-part-whole problems. In one type, the sizes of

both parts are given and the student is asked to find the size of the whole. In the other type, the size of one part and the size of the whole are provided and the student is asked to find the size of the other part.

| PART-PART-WHOLE  | EXAMPLE PROBLEM  |
|------------------|--|
| Whole Unknown    | Five boys and three girls are on the basketball team. How many children are on the basketball team?<br>$5 + 3 = \square$                         |
| One Part Unknown | Eight children are on the basketball team. Five are boys and the rest are girls. How many girls are on the basketball team?<br>$5 + \square = 8$ |

Part-part-whole problems involve a comparison of “parts” (subsets) with the “whole” (set). While the part-part-whole problems above could be modeled with manipulatives, the language in the problems does not suggest any action joining the “parts,” or subsets. This static relationship between subsets is the subtle difference that distinguishes these problems from join problems involving action.

### Compare Problems

Compare problems involve a comparison of two distinct, unconnected sets. Like part-part-whole problems, compare problems do not involve action. However, they differ from part-part-whole problems in that the relationship is not between sets and subsets but between two distinct sets. There are three types of compare problems, depending on which quantity is unknown: the difference (the quantity by which the larger set exceeds the smaller set), the quantity in the larger set, or the quantity in the smaller set. A relationship of *difference*, *more than*, or *less than* is found in compare problems.

| COMPARE                  | EXAMPLE PROBLEM   |
|--------------------------|---|
| Difference Unknown       | Ahmed has two brothers.<br>Christine has three brothers.<br>Christine has how many more brothers than Ahmed?<br>$3 - 2 = \square$ or $2 + \square = 3$      |
| Larger Quantity Unknown  | Ahmed has two brothers.<br>Christine has one more brother than Ahmed.<br>How many brothers does Christine have?<br>$2 + 1 = \square$                        |
| Smaller Quantity Unknown | Christine has one more brother than Ahmed.<br>Christine has three brothers.<br>How many brothers does Ahmed have?<br>$\square + 1 = 3$ or $3 - \square = 1$ |

Compare and part-part-whole problems exemplify that the operations of addition and subtraction are based on the relationships between two sets or between a set and its subsets. One reason to ask students to solve a variety of problem types is so that they will generalize the meaning of these operations beyond “actions” to relationships between sets.

Students' ability to solve the various problem categories is related to their ability to recognize the distinctions among them. Contexts and wording that indicate the actions or relationships in a problem can make a problem easier or more difficult. For example, consider these two problems:

*There are five boys and eight hats. How many more hats than boys are there?*

*There are five boys and eight hats. If each boy puts on a hat, how many hats are left over?*

Carpenter, Fennema, and Franke (1994, 10) found that the second problem is easier for students because the action is more explicit. The magnitude of the numbers in a problem also affects its level of difficulty. The following problems are both separate problems in which the change is unknown, but the second one is more difficult for young students because the numbers are larger:

*Liz had twelve pennies. She gave some pennies to Caitlin. Now she has eight pennies. How many pennies did Liz give Caitlin?*

*Liz had 45 pennies. She gave some pennies to Caitlin. Now she has 28 pennies. How many pennies did Liz give Caitlin?*

However, after students solve many separate/amount-of-change-unknown problems that involve small quantities, they are more likely to be able to answer similar problems with larger quantities.

Students' ability to translate the words in a problem to an operation that represents the relationships presented by the words takes time and many experiences. Teachers need to present all problem types throughout the school year so that students have the opportunity to develop meaning and fluency for word problems.

## Activity



### Classifying Addition and Subtraction Word Problems

*Objective: identify join, separate, part-part-whole, and compare problems.*

Before reading further, discuss with a colleague the four types of addition and subtraction problems. Next classify each of the following problems as join, separate, part-part-whole, or compare. Indicate which quantity is unknown and write a number sentence that represents the relationships expressed in each problem.

1. Carlton had three model cars. His father gave him four more. How many model cars does Carlton have now?
2. Juan has nine marbles. Mary has six marbles. How many more marbles does Juan have than Mary?
3. Janice has three stickers on her lunch box and four stickers on her book bag. How many stickers does she have in all?
4. Catherine had a bag of four gummy bears. Mike gave her some more. Now Catherine has seven gummy bears. How many gummy bears did Mike give her?
5. A third grader has seven textbooks. Four textbooks are in his desk. The rest of his textbooks are in his locker. How many textbooks are in his locker?
6. Vladimir had some baseball cards. Chris gave him 12 more. Now Vladimir has 49 baseball cards. How many baseball cards did Vladimir have before he received some from Chris?

7. Keisha had some crayons. She gave two crayons to Tanya. Now Keisha has nine crayons. How many crayons did Keisha have in the beginning?
8. Anthony had nine library books on his bookshelf. He returned six books to the library. How many books are left on his bookshelf?
9. There are nine board games in Joyce's room. Mariah has six fewer board games than Joyce. How many board games does Mariah have?
10. Eric weighed 200 pounds. During the summer, he lost some weight. Now he weighs 180 pounds. How many pounds did Eric lose?
11. Eli had some money. He gave his brother Johannes \$5.50. Now Eli has \$18.50 left. How much money did Eli have to begin with?
12. Graziella has four CDs. Fadia has eight more CDs than Graziella. How many CDs does Fadia have?

### **Things to Think About**

Sometimes it's difficult to distinguish part-part-whole problems from join problems and separate problems. The main difference is that in part-part-whole problems the relationship between entities is static (as in Question 3 above), whereas the relationships in join problems or separate problems are always described using combining or separating action verbs. The different addition and subtraction problems are not equally easy (or difficult) for students. In general, students find result-unknown problems in the join and separate categories, whole-unknown problems in the part-part-whole category, and difference-unknown problems in the compare category easier than problems in the remaining categories. Furthermore, the types of quantities used in problems can make them easier or more difficult; crayons are easy to model and count, whereas pounds are a more abstract quantity to represent.

Some of the problems (1 and 3, 8 and 9) can be represented by the same number sentence. You might want to write a few number sentences and then make up different types of problems that fit each sentence. This is another way to extend your own understanding of the different types of problems.

Three of the problems in this activity (1, 4, and 6) are "join" problems. In number 1, the result is unknown ( $3 + 4 = \square$ ), in number 4 the amount of change is unknown ( $4 + \square = 7$ ), and in number 6 the initial quantity is unknown ( $\square + 12 = 49$ ). There are four "separate" problems: 7, 8, 10, and 11. In number 8, the result is unknown ( $9 - 6 = \square$ ); in number 10, the amount of change is unknown ( $200 - \square = 180$ ); and in numbers 7 and 11, the initial quantity is unknown ( $\square - 2 = 9$  and  $\square - 5.50 = 18.50$ ). The two "part-part-whole" problems are 3 and 5. In number 3, the whole is unknown ( $3 + 4 = \square$ ), and in number 5, one of the parts is unknown ( $4 + \square = 7$ ). Finally, problems 2, 9, and 12 are "compare" problems. In number 2, the difference is unknown ( $9 - 6 = \square$ ); in number 9, the smaller quantity is unknown ( $9 - \square = 6$  or  $\square + 6 = 9$ ); and in number 12, the larger quantity is unknown ( $4 + 8 = \square$ ). ▲

## **2. Solution Strategies**

Young children use informal knowledge and different types of strategies to solve word problems and perform computations (Carpenter, Fennema, and Franke 1994; Carpenter, Fennema, Franke, Levi, and Empson 1999). Some of the strategies that students use are based on the structure and semantics of the problem. Other strategies are related to students' growing sense of number and their understanding of

mathematical properties and place value. In order to plan instruction that helps students develop increasingly sophisticated and efficient problem-solving strategies, it is important to recognize a variety of strategies and to understand how students' strategies develop.

Students tend to use three types of strategies when solving simple addition and subtraction word problems:

1. Strategies based on direct modeling with fingers or physical objects.
2. Strategies based on counting sequences.
3. Strategies based on number sense.

These strategies are hierarchical: students progress from modeling to counting to using number sense. However, this does not mean that students use only one strategy type for all problems; students apply different types of strategies to different types of problems. For example, when solving a separate word problem in which the amount of change is unknown, a student might use a modeling strategy. When solving a separate problem in which the result is unknown, the same student might use a counting strategy. And when solving a join problem involving numbers less than five in which the result is unknown, this student might use her knowledge of number facts, although she might revert to using less sophisticated strategies in join problems involving larger numbers.

### *Modeling Strategies*

In modeling strategies, students use physical objects such as blocks, counters, and fingers to model the actions and/or relationships in a problem. They then count some or all of these objects to obtain an answer. Many young students use modeling strategies when they first start to solve addition and subtraction problems. Older students unfamiliar with a particular problem type often use a modeling strategy to make sense of the problem. There are five common modeling strategies: joining all, separating from, separating to, adding on, and matching.

**Joining all.** In this addition strategy, students use physical objects to represent each of the addends in a problem. The answer to the problem is found by joining the sets of objects and counting them all, starting with one. Sometimes students first join the sets and then count all the items, sometimes they count one set followed by the other set. Interestingly, students don't seem to differentiate whether or not they physically join the sets, as long as they have modeled each set.

*Katie had two stickers. She bought five more stickers. How many stickers does Katie have now?*

Using objects or fingers, the student makes a set of 2 objects and a set of 5 objects. Then he counts the union of the two sets, starting with one.

**Separating from.** In this subtraction strategy, students use concrete objects to model the action of separating out the smaller quantity given in the problem. Usually the student counts the remaining objects to arrive at the answer.

*There were seven boys playing tag. Two boys went home. How many boys were still playing?*

Using objects or fingers, the student makes a set of 7 objects. She removes 2 objects. The number of remaining objects is the answer.

**Separating to.** This subtraction strategy is similar to the separating-from strategy except that objects are removed from the larger set until the number of objects remaining is equal to the smaller number given in the problem. Counting the number of objects removed provides the answer. This strategy involves some trial and error in that a student has to keep checking to see whether the appropriate amount still remains. Students often use this strategy to solve separate problems in which the change is unknown.

*There were 7 boys playing tag. Some went home. Now there are 2 boys playing tag. How many boys went home?*

The student counts out a set of 7 objects. Then he removes objects until only 2 remain. The number of objects removed is the answer.

**Adding on.** This strategy involves an addition action and is used by students to solve both addition and subtraction problems. A student sets out the number of objects equal to the smaller given number (an addend) and then adds objects one at a time until the new collection is equal to the larger given number. Counting the number of objects “added on” gives the answer. This strategy also involves some trial and error in that a student has to check regularly to see whether the larger number has been reached.

*Liz had two apples. Lyman gave her some more. Now Liz has five apples. How many apples did Lyman give her?*

The student makes a set of 2 objects. Then she adds objects to the set one at a time until there is a total of 5 objects. She finds the answer by counting the number of objects added.

**Matching.** This concrete strategy is used by many students to solve comparison problems in which the difference is unknown. The student puts out two sets of objects, each set representing one of the given numbers. The sets are matched one to one. Counting the objects without matches gives the answer.

*Tom has five brothers. Juan has two brothers. How many more brothers does Tom have than Juan?*

The student creates a sets of 5 objects and a set of 2 objects. He matches the objects in each set one to one and counts the number of unmatched objects.

### **Counting Strategies**

Counting strategies are more advanced solution processes than modeling strategies, because they are more abstract and there is more flexibility in which one to choose. The shift from modeling to counting depends on the development of certain number concepts and counting skills. Students must understand the relationship between counting and the number of elements in a given mathematical set (*cardinality*), must be able to begin counting at any number, and must be able to count backward. For



some situations, students must also be able to keep track of how many numbers they have counted and at the same time recognize when they have reached the appropriate number. Activities that focus on these counting skills and relationships will further students' ability to apply counting strategies to the solution of problems.

There are six common counting strategies: counting all, counting on from first, counting on from larger, counting down from, counting down to, and counting up from given (this list isn't exhaustive—don't be surprised if your students invent new ones!). In helping students use these strategies, it is important to realize that counting strategies are not mechanical techniques that students can simply memorize. Counting strategies are conceptually based and build directly on modeling strategies. Thus students need opportunities to connect modeling strategies with counting strategies. Many counting strategies have direct links to specific modeling strategies:

| MODELING STRATEGIES | COUNTING STRATEGIES     |
|---------------------|-------------------------|
| Joining all         | Counting all            |
| Separating from     | Counting down from      |
| Separating to       | Counting down to        |
| Adding on           | Counting up from given  |
| Matching            | None                    |
| None                | Counting on from first  |
| None                | Counting on from larger |

One way to help young students link strategies is to have them discuss how they got their solution. After students have presented their processes, teachers can highlight the similarities between two related strategies (e.g., separating from and counting down from). Furthermore, teachers can encourage the use of counting strategies and provide opportunities for students to practice them.

**Counting all.** This addition strategy is similar to the joining-all modeling strategy except that physical models or fingers are not used to represent the addends. As the name implies, students start the counting sequence with one and continue until the answer is reached. This strategy requires that students have a method of keeping track of the number of counting steps in order to know when to stop. Most students use their fingers to keep track of the number of counts. (Fingers here play a different role than in the joining-all modeling strategy; they are used to keep track of the number of steps rather than to model one of the addends.)

*Katie had two stickers. She bought five more stickers. How many stickers does Katie have now?*

The student begins the counting sequence with 1 for 2 counts (1, 2) and then continues on for 5 more counts (3, 4, 5, 6, 7). The answer is the last term in the counting sequence.

**Counting on from first.** With this addition strategy, the student recognizes that it is not necessary to reconstruct the entire counting sequence and begins "counting on" from the first addend in the problem.

*Katie had two stickers. She bought five more stickers. How many stickers does Katie have now?*

The student begins the counting sequence at 2 and continues on for 5 counts. The answer is the final number in the counting sequence.

**Counting on from larger.** This addition strategy is identical to the counting-on-from-first strategy except that counting begins from the larger of the two addends. This is a more sophisticated counting-on strategy, since implicit in its application is that the student understands that the order of the addends does not matter in addition problems.

*Katie had two stickers. She bought five more stickers. How many stickers does Katie have now?*

The student begins the counting sequence at 5 and continues on for 2 counts. The answer is the final number in the counting sequence.

**Counting down from.** This subtraction strategy is the parallel counting strategy to the separating-from modeling strategy. In this strategy students initiate a backward counting sequence beginning at the given larger number. The counting sequence contains as many numbers as the given smaller number.

*There were seven boys playing tag. Two boys went home. How many boys were still playing?*

The student begins a backward counting sequence at 7. She continues the sequence for 2 counts (i.e., 6, 5). The last number in the counting sequence (5) is the answer.

**Counting down to.** This subtraction counting strategy is parallel to the separating-to modeling strategy. Students use a backward counting sequence until the smaller number is reached. How many numbers there are in the counting sequence is the solution. Students often use their fingers to keep track of the counts, but they are not actually modeling the situation.

*There were seven children playing tag. Some went home. Now there are two children playing tag. How many children went home?*

The student starts a backward counting sequence at 7 and continues until 2 is reached (i.e., 6, 5, 4, 3, 2). The answer is how many numbers there are in the counting sequence (5).

**Counting up from given.** This counting strategy is parallel to the adding-on modeling strategy. The student initiates a forward counting strategy from the smaller number given. The sequence ends with the larger number given. The student keeps track (often using his or her fingers) of how many numbers there are in sequence.

*Liz had two apples. Lyman gave her some more. Now Liz has five apples. How many apples did Lyman give her?*

The student starts counting at 2 and continues until 5 is reached (i.e., 3, 4, 5). The answer is how many numbers there are in the sequence (3).

## Activity



### Matching Problems and Strategies

*Objective: link problem types with strategies that students typically use in solving them.*

Understanding the relationships within problems will help you link problem structure to students' solution strategies. Analyze each problem in Activity 1 in terms of which strategies could be used to model the actions or relationships in the problem. Indicate both a modeling strategy (with objects) and a counting strategy. Many problems can be solved using a number of different strategies or by using a strategy not presented here.

### Things to Think About

Students' strategy choices are influenced by a number of factors. First, students are most likely to pick a solution strategy that matches the structure of a problem. Whether the strategy chosen is a modeling, counting, or number sense strategy depends in part on students' familiarity with the type of problem (can they make sense of which operation to use?) and on students' understanding of number and counting. When students have not made sense of the relationships in a problem and have not yet connected these relationships to specific operations, they are more likely to use a modeling strategy.

Problems are much more difficult if students do not have a process available to model the actions or relationships. For example, a compare problem in which the difference is unknown is difficult if students have never considered or seen a matching strategy. You can help students learn new strategies by creating specific problems for them to solve and then having them discuss their various solution strategies in pairs and as a whole class. New approaches and strategies are often introduced this way. While occasionally you may wish to model a solution strategy for students, it is important that students don't just observe the strategy but use and discuss it. Students also need many opportunities to apply new strategies. After a strategy has been introduced, you should assign problems that enable students to practice and refine the particular strategy.

Some strategies are not as widely used as others (counting down is not used as much as counting up from given, for example), and some students never use some of the strategies. In many cases, students don't even differentiate between strategies (some look at counting on and counting up from given as the same strategy). Likewise, as students mature, they often change which strategies they use (the matching strategy is abandoned after the early grades, for example).

Finally, remember that it is not necessary for students to label their solution strategies (e.g., counting on). But it is important for you to recognize the different types of strategies in order to choose future instructional activities that support students' developing proficiencies.

Here are the most likely ways to solve the problems in Activity 1:

1. Counting all; counting on from larger.
2. Matching; counting up from given.
3. Counting all; counting on from larger.
4. Adding on; counting up from given.
5. Separating from; counting down from.

6. Adding on; counting up.
  7. Counting all; counting on from larger.
  8. Separating from; counting down from.
  9. Separating from; counting down from.
  10. Separating to; counting down to.
  11. Counting all; counting on from larger.
  12. Separating from; counting down from. ▲
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### *Number Sense Strategies*

Students eventually replace modeling and counting strategies with number sense strategies. To do so they must (1) understand whether the relationships and actions in a problem require them to add or subtract, and (2) be able either to recall number facts or to use known number facts to derive new facts.

The relationship between counting and mental strategies is not clear. We do know that students' previous use of counting strategies helps them recall number facts. Certainly, the ability to move to this level of abstraction depends in part on being able to make sense of the relationships in many types of problems, and this means students must have solved many different kinds of problems.

Being able to use known number facts to derive new facts is linked to an understanding of part-whole relationships (not to be confused with the part-part-whole problem type described in Section 1). Quantities can be interpreted as comprising other numbers. For example, a set of eight objects can be represented using two or more parts such as 0 and 8; 1 and 7; 2, 2, and 4; eight 1s; and so on. Eight is the "whole," the numbers making up eight are the "parts." If students do not understand number in terms of part-whole relationships, they are not able to decompose wholes and recombine parts flexibly. For example, to find the answer to  $6 + 8$ , some students decompose 8 into two parts (6 and 2) and reinterpret the calculation as  $6 + 6 + 2$ . If they already know that  $6 + 6 = 12$ , they can use that information to determine that  $6 + 8 = 14$ . Some researchers believe that the most important conceptual achievement in the early grades occurs when students interpret number in terms of part-whole relationships. It must be emphasized that the use of part-whole reasoning to find the answers to basic facts is not limited to superior students.

Students' solutions involving part-whole relationships are often based on number facts that sum to 10. For example, a student might calculate  $4 + 7$  like this: *three plus seven is ten, and four plus seven is just one more, so the answer is eleven*. This student knows that the number fact for  $3 + 7$  is 10 and understands that 4 can be thought of as 3 and 1 even though that relationship is not explicitly stated. Doubles facts are also used by many students to derive new facts. For example, to solve  $7 + 8$  a student might respond: *eight plus eight is sixteen, and seven plus eight is just one less than sixteen, so the answer is fifteen*.

Other factors that contribute to students' ability to use mental strategies based on number sense are their knowledge and understanding of addition and subtraction properties and relationships. Knowing about the commutative property, which states that the order of the addends (e.g.,  $9 + 4$  or  $4 + 9$ ) does not affect the sum, lets students transpose a problem to an easier form. The inverse relationship between addition and subtraction can also help students find answers (e.g., knowing that  $6 + 7 = 13$ , it follows that  $13 - 6 = 7$ ) or develop new strategies (e.g., counting up).

The time needed before students consistently use mental strategies to solve problems varies, but for some students it can take a number of years. Mathematical tasks that highlight part-whole relationships in general and that focus on doubling and numbers summing to ten in particular provide a foundation on which students can build. Likewise, instruction that highlights the relationship between addition and subtraction and enables students to reflect on properties also contributes to children's number and operation sense.

## Activity



### Observing How Students Solve Problems

*Objective: use knowledge of problem types and solution strategies to understand students' thinking.*

Write one or two problems that fit the classification scheme described in Section 1 and are appropriate for the students you teach. (You may want to begin with join problems in which the result is unknown and separate problems in which the amount of change is unknown.) In individual five-minute interviews, ask students to solve the problems. Record each student's solution strategy for each problem. When you have all your data, group students' solution strategies by type. How many students are using modeling, counting, or number sense strategies? How might you use this information to further students' understanding?

#### **Things to Think About**

When watching students solve problems, it isn't always easy to determine what strategy they are using, since they may be unable to articulate their thinking. You may have to ask probing questions or watch for overt behavior in order to hypothesize about students' thought processes. Furthermore, when a child uses his fingers, is he using a modeling strategy or a counting strategy? It's a modeling strategy if the fingers are used to show the quantities in the problem; it's a counting strategy if the fingers are used to keep track of how many numbers are in a counting sequence.

You may find that students use different strategies for join problems in which the result is unknown than they do for separate problems in which the amount of change is unknown. This may be partly because they haven't had much experience with the latter type of problem.

If students are using modeling strategies, you may wish to include mathematical tasks and sequences in your instruction that support the development of counting strategies. Providing students with many opportunities to practice counting forward and backward (especially starting in the middle of a sequence), identify patterns when counting (e.g., when we count by 100s, what changes with each increase? what doesn't change with each increase?), and connect a modeling strategy with a counting strategy will help. If students are using any of the counting strategies, you can focus on instruction that supports the use of more sophisticated methods. Activities that enable students to grapple with part-whole relationships; that emphasize number combination strategies such as doubles, doubles plus one, and doubles minus one (e.g.,  $5 + 6$  can be thought of as  $5 + 5 + 1$  or as  $6 + 6 - 1$ ); and that have students exploring the relationship between addition and subtraction will all contribute to your goal. ▲