This instruction might include activities that ask students to explore relationships between operations (e.g., addition and subtraction) or to explore properties and then apply the properties to learning other facts. For example, students might determine 8 × 7 by decomposing 7 into 3 + 4 and using their knowledge of 8 × 3 and 8 × 4 (24 + 32 = 56). This strategy works because of the distributive property (8 × (3 + 4)). Likewise, students can further their knowledge by writing word problems that use the facts in a meaningful context, using concrete objects to connect these situations with symbols, and discussing relationships within fact families.

Learning facts is a gradual process that for most students takes a number of years. (See Chapter 3 for specifics on how skills develop in addition and subtraction.) Eventually, however, students need to commit the basic facts to memory. Naturally, the exact age when a particular student masters these facts varies. In general, however, most students have mastered addition/subtraction facts by the end of third grade and multiplication/division facts by the end of fifth grade. "Mastery" does not imply that students are human calculators able to perform at lightning speed. It means that they know the facts well enough to be efficient and accurate in other calculations.

3. Algorithms

Algorithms, as the term is applied to the arithmetic procedures students traditionally have learned in school, are systematic, step-by-step procedures used to find the solution to a computation accurately, reliably, and quickly. Algorithms, whether performed mentally or with paper and pencil, a calculator, or a computer, are used when an exact answer is required, when an estimate won't suffice. Because they are generalizations that enable us to solve classes of problems, they are very powerful: we can solve many similar tasks (1,345,678 − 987,654 and 134 − 98, for example) using one process. In the best of circumstances, algorithms free up some of our mental capacity so that we can focus on interpreting and understanding a solution in the context of a problem. In the worst of circumstances, algorithms are used when a task could be done mentally or are applied by rote with little understanding of the bigger mathematical picture—why the calculation is important and how the answer will be used.

There are many different algorithms for performing operations with numbers. Some of these algorithms are now referred to as *standard* or *conventional* simply because they have been taught in the majority of U.S. classrooms over the past fifty years. For example, you may have learned (or taught) the standard addition algorithm shown below, in which you "carry" from the ones column to the tens column to the hundreds column:

$$\begin{array}{r} {\scriptstyle 11} \\ 456 \\ +899 \\ \hline 1355 \end{array}$$

Interestingly, some of the conventional algorithms in the United States are not the standard algorithms in Europe or South America. Children around the world learn different computational procedures in school.

Other algorithms are known as *alternative* algorithms—they differ from the standard algorithms for adding, subtracting, multiplying, and dividing. Alternative algorithms also are accurate, reliable, and fast. Alternative algorithms such as the lattice method for multiplication have sometimes been used in schools as enrichment activities. Today many alternative algorithms are part of the elementary mathematics curriculum. Making sense of algorithms can be instructive; students figure out why certain procedures work, which leads them to insights into important ideas such as place value and the distributive property of multiplication over addition.

In the 1990s many researchers and mathematics educators began to question the wisdom of the rote teaching of conventional algorithms to students in the elementary grades. Research has shown that when children simply memorize the steps to complete the standard addition and subtraction algorithms, they lose conceptual understanding of place value. In contrast, students who invent their own procedures or algorithms for solving addition and subtraction problems have a much better understanding of place value and produce more accurate solutions (Kamii 1994; Kamii and Dominick 1998; Narode, Board, and Davenport 1993). Many mathematics educators now suggest that instead of teaching students standard algorithms as the only or best ways to compute with paper and pencil, we provide many opportunities for students to develop, use, and discuss a variety of methods. Having students invent algorithms leads to enhanced number and operation sense as well as flexible thinking (Burns 1994b; Carroll and Porter 1997, 1998).

Many elementary curriculums are designed so that children initially use logical reasoning and their understanding of number (e.g., that 36 can be decomposed into 30 + 6), place value, and mathematical properties to invent their own algorithms and procedures to solve addition and subtraction problems. The purpose of this type of instruction is to extend and expand students' understanding of number, place value, decomposition, and recomposition (see Chapter 1 for elaboration) as they learn to compute. Student-invented or student-generated procedures sometimes are algorithms; that is, they can be generalized to classes of problems and they enable the student to produce accurate answers. (They many not be efficient or easy to use, however.) Other procedures are not algorithms; they may enable a child to calculate a correct answer, but they cannot be generalized to other problems.

However, as students progress through the grades, they need to acquire efficient ways to compute, and student-generated methods may not suffice. Thus, after students have experimented with developing their own methods, teachers often introduce standard and alternative algorithms as a focus of study. When these algorithms are examined and analyzed, not taught in a rote way, students have the opportunity to build on their already established understanding of number and place value to expand their repertoire of efficient, reliable, and generalizable methods.

In schools we often associate the study of algorithms with paper-and-pencil procedures. One useful by-product of paper-and-pencil algorithms is that they provide a written record of the processes used to solve a problem. Students can use this record to refine procedures, share what has been accomplished, and reflect on solutions. Keeping a record of the steps in an algorithm is especially important when students are trying to make sense of the reasoning involved in the computation. The activities in this section examine a variety of algorithms and student-invented procedures for whole number computations. The goal is for you to understand why these methods work and to consider the mathematics that students have made sense of in order to use them.

Activity

Analyzing Students' Thinking, Addition

Objective: learn some common addition strategies.

Examine the following examples of students' procedures for solving addition problems. First, explain what the student did to obtain a correct answer. Then use the student's algorithm to solve the problem 1367 + 498.

Kelly	*Rudy*	*Andy*	
567	567	567 ⇒	600
+ 259	+ 259	+ 259	+ 226
700			$\boxed{826}$
110	200 – 567, 667, 767		
16	50 – 777, 787, 797, 807, 817	259	
$\boxed{826}$	9 – 818, 819, 820, 821, 822	– 33	
	823, 824, 825, $\boxed{826}$	226	

Things to Think About

Kelly's algorithm is sometimes called the partial sums method. She added the digits in the problem by place value, starting with the largest place value (hundreds)—500 + 200 = 700, 60 + 50 = 110, 7 + 9 = 16. After calculating the partial sums, Kelly added them (700 + 110 + 16 = 826). Kelly is able to decompose numbers into hundreds, tens, and ones, add like units, and then recompose the three subtotals to produce the final sum. Since her use of this algorithm implies that she understands place value through hundreds, she would probably be able to generalize this approach to four-digit and larger sums. To solve 1367 + 498 using Kelly's method you would figure this way:

$$\begin{array}{r} 1367 \\ +\ 498 \\ \hline 1000 \\ 700 \\ 150 \\ 15 \\ \hline 1865 \end{array}$$

How did Rudy solve 567 + 259? It appears that he started with 567 and either counted on or added on, first by hundreds, then by tens, finally by ones. He started with the hundreds, writing 567, 667, 767. Then he continued with five tens: 777, 787, 797, 807, 817. Finally he finished with the ones: 818, 819, 820, 821, 822, 823, 824, 825, 826. While this procedure works, it is prone to error, especially when an increase in one grouping necessitates an increase in the next one up (797 to 807, for example). To solve 1367 + 498 using Rudy's method, round 498 up to 500, count on by hundreds (1467, 1567, 1667, 1767, 1867), and then count backward by ones (1866, 1865). Sometimes students use an invented procedure for a short period of time and then move on to another, more efficient procedure or algorithm. The importance of classroom discussion about solution procedures cannot be overemphasized—students often learn about other approaches when their classmates describe their method.

Andy used an approach different from the other two students in that he did not decompose the numbers based on hundreds, tens, and ones. Instead, he changed both numbers to other numbers that he thought would be easier to use. He started by adding 33 to 567 to get 600. He then subtracted the 33 from 259 to get 226. Finally, he added his two adjusted numbers: 600 + 226 = 826. Andy's method works because of the associative and commutative properties. The identity property of addition also comes into play: adding zero—33 + (−33) = 0—doesn't change the sum:

$$\begin{aligned} 567 + 259 &= (567 + 259) + (33 + {}^{-}33) \\ &= (567 + 33) + (259 + {}^{-}33) \\ &= (567 + 33) + (259 - 33) \\ &= 600 + 226 \\ &= 826 \end{aligned}$$

Andy's procedure is very efficient with an addition such as 1367 + 498, because it's easy to "see" how to change 498 to 500 and compensate by subtracting two from 1367. However, this approach may not be all that ideal with problems such as 2418 + 1725 (though it will work), since the additions and subtractions leading to the adjusted numbers may require a lot of mental energy. ▲

Activity

Analyzing Students' Thinking, Subtraction

Objective: learn some common subtraction strategies.

Examine the following examples of students' procedures for solving the same subtraction problem. What did each student do to obtain a correct answer? Why does the student's algorithm work?

Caitlin

$$\begin{array}{r l} 63 & \\ -\ 18 & \quad 10 + 8 \\ \hline 63 - 10 = 53 & \\ \hline 52, 51, 50, 49, & \\ 48, 47, 46, 45 & \\ & \\ \boxed{45} & \end{array}$$

Louis

$$\begin{array}{r} 6\overset{13}{\cancel{3}} \\ -{}^{2}\cancel{1}8 \\ \hline \boxed{45} \end{array}$$

Kenley

$$\begin{array}{r l l r} 63 & & & \\ -18 & & & \\ \hline 28 & \left.\begin{array}{l} 10 \\ 10 \\ 10 \\ 10 \end{array}\right\} & & \\ 38 & & & \\ 48 & & & \\ 58 & & \text{or} & 40 \\ +5 & & & 5 \\ \hline & & & \boxed{45} \end{array}$$

Things to Think About

Caitlin solved this computation by inventing a procedure based on her understanding of place value and counting back. First she decomposed 18 into 10 + 8, then she removed the 10 from 63 (63 − 10 = 53), and finally she counted back 8 from 53 to 45. This procedure is commonly seen in second-grade classrooms in which children have been encouraged to invent their own procedures that make sense to them. Let's try Caitlin's procedure with 152 − 39.

$$\begin{array}{r l} 152 & \\ -\ 39 & \quad 30 + 9 \\ \hline 100 & \\ 52 - 30 = 22 & \\ 122 & \\ \hline \end{array}$$

$$122, 121, 120, 119, 118, 117, 116, 115, 114, \boxed{113}$$

While it is unlikely that you would use this procedure for larger multidigit subtraction problems, adults do use a combination of methods including counting back with simple calculations such as determining elapsed time (e.g., *for how many hours do I pay the baby-sitter if it is now 12:45 A.M. and I left at 4:30 P.M.?*). Depending on the context and the numbers in a problem, simple counting procedures can be quick and efficient.

Louis moved to the United States when he was eleven and learned this algorithm in school in Italy. What are the steps in Louis's algorithm? First, Louis noticed that there were not enough ones in the ones place (3) to subtract 8, so he changed the 3 to a 13. Because he added 10 to the 63 (63 + 10 = 60 + 13), he had to add 10 to the 18 (10 + 18 = 28) in order not to change the problem. Louis recorded the 10 added to the 18 by changing the 1 in the tens place to a 2. He then subtracted the ones (13 − 8 = 5) and the tens (6 − 2 = 4) to obtain the answer of 45. This algorithm uses compensation (see page 37 in this chapter)—if you add (or subtract) the same number to both the minuend and the subtrahend in a subtraction problem, the difference is not affected.

$$\begin{array}{r} 63+10 \\ -\ 18+10 \\ \hline 45 \end{array} \;\Rightarrow\; \begin{array}{r} 73 \\ -\ 28 \\ \hline 45 \end{array} \qquad \begin{array}{r} 63+10= \\ -18+10= \\ \hline \\ \end{array} \; \begin{array}{r} 60+13 \\ 20+\ 8 \\ \hline 40+\ 5 \end{array}$$

What is interesting about Louis's algorithm is that the 10 is added to different place values, which simplifies the computation. Let's try his procedure with 81 − 58:

$$\begin{array}{r} 8\overset{11}{\cancel{1}} \\ -\ {}^{6}\cancel{5}8 \\ \hline 23 \end{array}$$

Kenley understands the inverse relationship between addition and subtraction and used an approach sometimes referred to as "adding on" or "counting up." It is often used by students for subtraction word problems involving money when making change. For example, to solve the problem, "How much change should you receive from a $5.88 purchase if you give the clerk a $10 bill?," you can add on from $5.88 to $10.00 ($.12 makes $6.00 plus $4.00 makes $10.00, the change is $4.12). The amount Kenley added on to 18 to reach 63 was her answer. She counted by tens until she was close to but not above 63: 28, 38, 48, 58. She recorded that she had added four tens, or 40. When she realized that five more ones would bring her to 63 (58 + 5 = 63), she had her final answer of 45. Let's use Kenley's method to solve 81 − 58:

$$\begin{array}{r} 81 \\ -\ 58 \\ \hline \end{array} \qquad \begin{array}{r} 58 \\ +\boxed{20} \\ \hline 78 \\ +\ \boxed{3} \\ \hline 81 \end{array} \quad \begin{array}{c} 68,\ 78 \\ \Leftarrow \\ 79,\ 80,\ 81 \\ \Leftarrow \\ \\ \end{array} \quad \boxed{23}$$

The standard subtraction algorithm is the one that most adults were taught when students. It is an efficient method that is based on decomposition and regrouping. Let's examine how Juan solves 152 − 39.

$$\begin{array}{r} 1\overset{4}{\cancel{5}}\overset{12}{\cancel{2}} \\ -\ 39 \\ \hline 113 \end{array}$$

Juan's actions are based on the fact that numbers can be decomposed into hundreds, tens, and ones (152 = 100 + 50 + 2) and recomposed in less efficient groupings (152 = 100 + 40 + 12) that can be used in certain types of problems. He notes that since there are only 2 ones in the ones place and he wants to subtract 9 ones, he needs to regroup 1 ten from the tens place and combine these 10 ones with the 2 ones for a total of 12 ones. This leaves only 4 tens in the tens place. Now he can subtract by place value (12 − 9 = 3, 40 − 30 = 10, 100 − 0 = 100). In the most abbreviated version of this algorithm, Juan subtracts 4 − 3 and 1 − 0 as if these numbers represented ones, but knows that by their placement in the tens and hundreds columns, respectfully, they represent 40 − 30 and 100 − 0. As with many algorithms, this one can be applied by rote. If students solve problems such as this one using a memorized procedure, they usually are unable to explain their actions. For example, they can't tell you what the 4 or the 12 above the tens and ones places represent and why those numbers are being used. Students who are unclear on the purpose of regrouping sometimes record answers that make no sense. Or, if students report that "this is how you do it" when asked to explain their steps, it is a signal that they may have memorized the procedures to execute the algorithm without understanding them. However, if students understand the relationships involved, they will have a solid grasp of place value and decomposition/recomposition and will find this algorithm very useful.

The application of the standard algorithm is more difficult when the minuend contains one or more zeroes. Teachers and students alike agree that in these situations the standard algorithm is confusing and hard to use! The regrouping process involves many steps and requires that students understand the relationships between larger place values. Let's take a look at 5009 − 836.

$$\begin{array}{r} 9 \\ 4\,\cancel{10}\,10 \\ \cancel{5}\cancel{0}\cancel{0}9 \\ -\ 836 \\ \hline 4173 \end{array}$$

When we decompose the number 5009 into place values, there are no tens or hundreds to regroup in different ways. This means that we must decompose the 5000 into 4000 + 1000 and regroup the 1000 as 10 hundreds. This is shown in the algorithm by crossing out the 5 (5 thousand), replacing it with a 4 (4 thousand) and putting a 10 over the hundreds to indicate the regrouping of 1000 as 10 hundreds. Then we decompose the 10 hundreds into 900 + 100 and regroup the 100 as 10 tens. Using symbols, this is recorded by crossing out the 10 above the hundreds place and writing a 9 (9 hundred) and placing a 10 above the tens place (10 tens). Finally, we are ready to subtract place by place (9 − 6 ones, 10 − 3 tens, 9 − 8 hundreds, and 4 − 0 thousands).

Whereas the standard algorithm can be quite efficient, some students find it easier to use compensation when subtraction involves numerous zeroes in the minuend. For example, subtract 9 from both numbers (5009 − 9 and 836 − 9). The equivalent subtraction problem of 5000 − 827 can be solved in a variety of ways. Using Kenley's "adding on" method, students might count on from 827 starting at the ones place (827 ⇒ 3 + 70 + 100 + 4000). Or they might add 173 to both numbers (5000 + 173 = 5173, 837 + 173 =1000) and perform the simpler calculation, 5173 − 1000 = 4173. Take a minute and think about what students need to understand in order to be able to use a variety of solution methods. ▲