The operator interpretation of fraction can also be used to solve this problem. If we divide all 10 fish into thirds, all 14 fish into fourths, and all 16 fish into fifths, each unit fraction can operate on the total number of fish. For example, multiplying $\frac{1}{3}$ by 10 gives a mixed number, $\frac{10}{3}$, or $3\frac{1}{3}$; multiplying $\frac{1}{4}$ by 14 gives $\frac{14}{4}$, or $3\frac{2}{4}$; and multiplying 16 by $\frac{1}{5}$ results in $\frac{16}{5}$, or $3\frac{1}{5}$ fish per person.

10 fish, $\frac{1}{3}$ per person
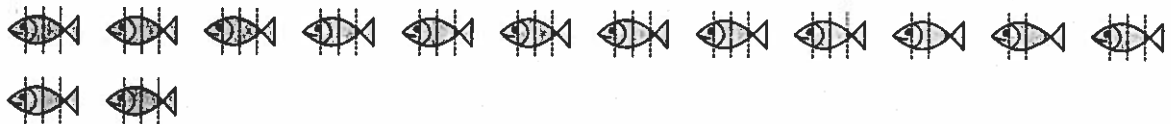
14 fish, $\frac{1}{4}$ per person
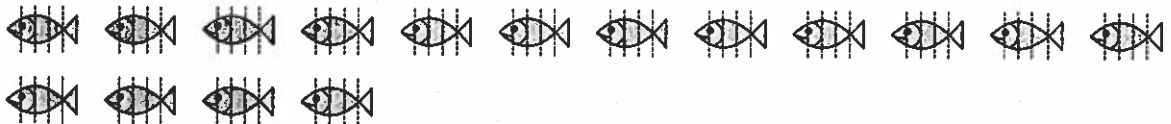
16 fish, $\frac{1}{5}$ per person

▲

2. Equivalence and Ordering

Equivalence is one of the most important mathematical ideas for students to understand, particularly with regard to fractions. Equivalence is used when comparing fractions, ordering fractions, and adding and subtracting fractions. Equivalent fractions are fractions that represent equal value; they are numerals that name the same fractional number. When represented using a number line, equivalent fractions represent the same distance. Equivalent fractions are obtained when both the numerator and the denominator of a fraction are either multiplied or divided by the same number:

$$\frac{a}{b} = \frac{a \times c}{b \times c} = \frac{ac}{bc} \quad \text{or} \quad \frac{ac}{bc} = \frac{ac \div c}{bc \div c} = \frac{a}{b}$$

These relationships are illustrated in the diagrams on page 115, using the parts-of-a-whole interpretation of fraction. The shaded portion of Figure 1 shows $\frac{3}{8}$. Figure 2 was created by dividing each of the eighths in half—we can also say that the number of shaded regions is doubled, as is the total number of regions:

$$\frac{3}{8} = \frac{3 \times 2}{8 \times 2} = \frac{6}{16}$$

If each eighth was instead cut into four pieces, another equivalent fraction would be obtained (see Figure 3):

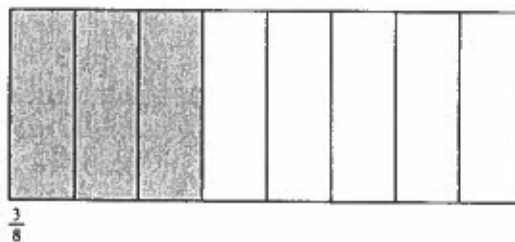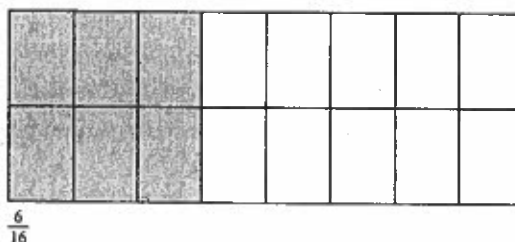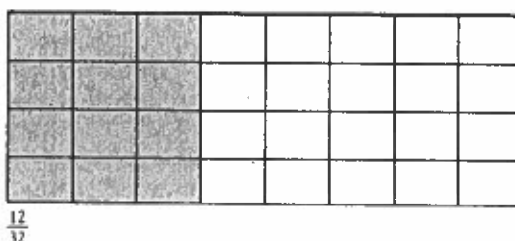$$\frac{3}{8} = \frac{3 \times 4}{8 \times 4} = \frac{12}{32}$$

Figure 1

$\frac{3}{8}$

Figure 2

$\frac{6}{16}$

Figure 3

$\frac{12}{32}$

Children often use a doubling strategy to form equivalent fractions. This can be an effective strategy in some situations, as shown in Figures 1 through 3 ($\frac{3}{8} = \frac{6}{16} = \frac{12}{32}$), but notice that doubling does not produce all equivalent fractions ($\frac{9}{24}$ and $\frac{15}{40}$ to name a few).

One underlying assumption regarding equivalent fractions is that when we state that two fractions are equivalent, this implies the "wholes" are the same size. However, students do not always recognize this important fact. Furthermore, students sometimes do not focus on the areas covered by equivalent fractional amounts but instead count the number of pieces, incorrectly assuming that $\frac{3}{8}$ is not equivalent to $\frac{6}{16}$ because $3 \neq 6$. Instructional tasks that focus on equivalence need to direct students' attention to whether or not the "wholes" are the same size and whether the fractional amounts, distances on a number line, or areas in each of the wholes are the same.

Encountering a variety of instructional models and applications may help students generalize some of the key ideas about equivalent fractions. In particular, the idea that multiplication and division can be used to form equivalent fractions needs to be examined in different situations with the different interpretations of fractions. For example, number lines can be used to show that $\frac{3}{4}$ is equivalent to $\frac{15}{20}$; both fractions represent the same distance though on the second number line each fourth is divided into five sections in order to create twentieths.
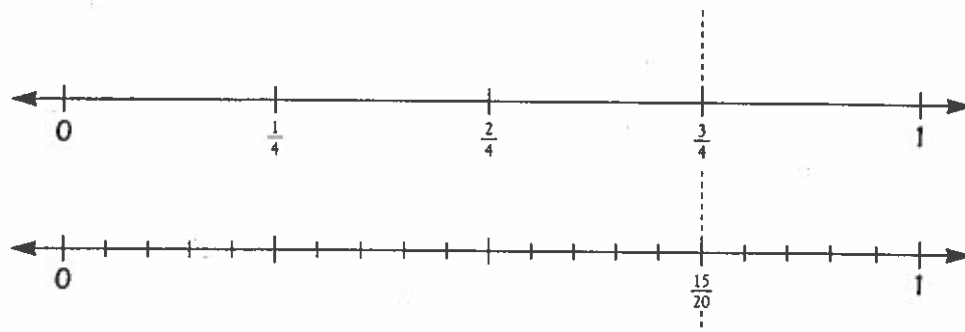
Dividing each fourth into five equal parts is equivalent to multiplying the numerator and denominator of $\frac{3}{4}$ by 5.

Using the operator interpretation, students might explore how $\frac{2}{3}$ of 9 is equivalent to $\frac{6}{9}$ of 9 using objects or pictures. One way to think of $\frac{2}{3}$ of 9 is to divide the 9 apples below into 3 groups, with 3 apples in each. Two groups of 3 is 6.

$\frac{2}{3}$ of 9 = 6
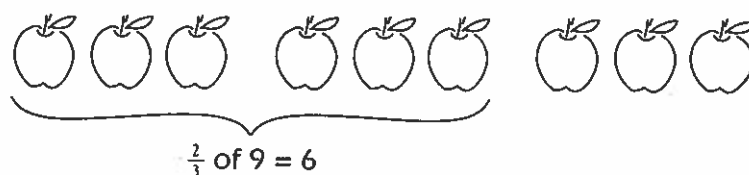
Likewise, $\frac{6}{9}$ of 9 also equals 6. Since $\frac{2}{3} = \frac{6}{9}$, operating (in fact, multiplying 9 by the fraction) with either of these fractions results in the same product, 6.

$\frac{6}{9}$ of 9 = 6
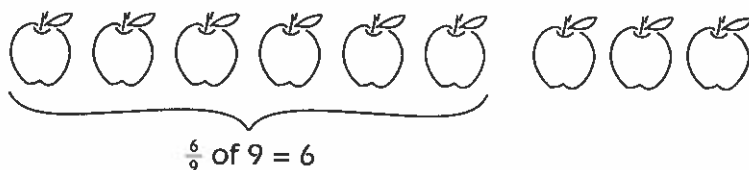
Pattern blocks can also be used to explore the multiplicative relationships in equivalence. If the yellow hexagon from a set of pattern blocks represents one "whole," then a blue rhombus represents one third and two green triangles represent two sixths. Students can construct models of these fractions and then use the models to describe how sixths can be grouped to form thirds or how thirds can be divided to form sixths.

sixths   →   thirds   thirds   →   sixths
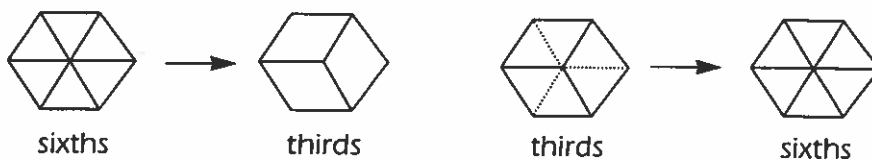
Children deal with equivalence informally when counting money, using measuring cups, telling time, folding paper, sharing snacks, and eating pizza. Thus, many children are familiar with the general idea of equivalency long before the concept is introduced in school. Why then do children have such difficulty with this important idea? In part it is because we sometimes forget to design instruction around children's prior knowledge—we jump into a topic as if they know nothing or know everything! In addition, some textbooks devote very little time to exploring the meaning of equivalent fractions and instead focus on symbolic manipulation (having students practice finding common denominators, for example). Instructional activities that use models and drawings and ask students to reflect on why two fractions are or are not equivalent are necessary. The activities in this section highlight important ideas related to equivalence and order.

Activity

Exploring Fourths on a Geoboard

Objective: explore the idea that equal fractions, using a part-whole model, do not have to have the same shape but must have equal area.

Materials: a geoboard or geoboard dot paper.

On a geoboard (or geoboard dot paper) make the largest square possible. Now divide the square to show fourths. Each fourth must be an area that if cut out of paper would remain in one piece. Make another square and show fourths that are irregularly shaped. Next make fourths that are not congruent but have equal area. Take each of your previous drawings of fourths and divide them further to show eighths.
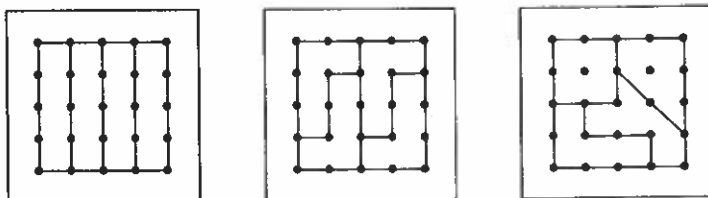
Things to Think About

What strategies did you use to make fourths? Since there are 16 square units in the square, each fourth must cover exactly 4 square units. Textbooks often unwittingly create misconceptions about fractions by presenting pictures of "wholes" partitioned into identical fractional parts. In this case, where there are 16 square units in total, any configuration that uses 4 square units is one fourth of the total area. The idea that equal fractional pieces don't have to be identical in shape but simply must cover the same area or space is an important one: it's part of the foundation on which other equivalence relationships are built. For example, a rectangular-shaped $\frac{1}{4}$ of the geoboard covers the same area as an irregularly shaped $\frac{1}{4}$ of the geoboard (4 square units), but the $\frac{1}{4}$s do not look identical. Students must overcome their tendency to rely on visualization for verification of equality and instead consider the relationships established by dividing a whole into n number of parts.

The geoboard also can be used to show why $\frac{1}{4} = \frac{2}{8}$. Although these fractions cover an equivalent area, $\frac{2}{8}$ consists of more but smaller fractional pieces. This can be demonstrated by cutting each fourth of the geoboard into two pieces. Notice that this is represented symbolically by multiplying by one ($\frac{2}{2}$):

$$\frac{1 \times 2}{4 \times 2} = \frac{2}{8} \qquad \blacktriangle$$

Activity

Tangram Fractions

Objective: use tangram puzzle pieces to find equal but noncongruent areas that have equivalent representations in terms of fourths, eighths, and sixteenths.

Materials: a tangram puzzle.

Use a tangram puzzle or the drawing below and determine the fractional values of each of the pieces. The complete tangram square is the unit, or whole. Which of the pieces have equal area? Think about how you can identify the fractional value of each piece. What role will equivalent fractions play in your decisions? Label each piece using denominators of fourths, eighths, and sixteenths.
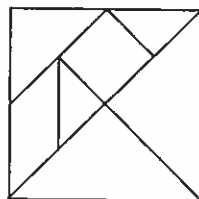
Things to Think About

There are a number of ways to determine the values of the tangram pieces. Since the two large triangles cover one half of the whole square, each large triangle represents $\frac{1}{4}$. Imagine covering the other half of the tangram square with small triangles. Eight small triangles cover one half of the whole, so sixteen of these small triangles cover the whole square. Thus, a small triangle is $\frac{1}{16}$ of the whole. The square, the medium triangle, and the parallelogram each cover the same area as two small triangles; each has a fractional value of $\frac{2}{16}$, or $\frac{1}{8}$. How might you demonstrate that $\frac{1}{4}$ is equivalent to $\frac{4}{16}$? Try covering the large triangle with the medium triangle and the two small triangles. Other equivalencies, such as $\frac{2}{8} = \frac{1}{4}$, can be illustrated using two medium triangles, which are the same size as one large triangle.

$\blacktriangle$
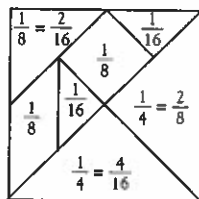
When we compare two fractions or order three or more fractions, we apply ideas involving equivalence. For example, if you are given two fractions to compare, what steps do you take? Most adults first apply rules they have learned to form equivalent fractions with common denominators, and then they compare these equivalent fractions. They don't first think about the fractions in terms of their relative size—that is, they don't reason about how large or small these fractions are until after they have equivalent fractions with common denominators. However, informal methods that focus on understanding the size of a fraction can initially be used to compare and order fractions in conjunction with formal procedures such as finding common denominators. The next three activities ask you to explore some informal but efficient methods for comparing and ordering fractions.

Activity

Numerators and Denominators

Objective: use relationships among common numerators and common denominators to compare and order fractions.

How is the size of a fraction related to the numerator and denominator of the fraction? More specifically, how is the size of $\frac{2}{4}$ changed when you increase the numerator by one—to $\frac{3}{4}$? How is the size of $\frac{2}{4}$ changed when you increase the denominator by one—to $\frac{2}{5}$? Pick a common fraction. Increase the numerator by one and write the resulting fraction. Use the same initial fraction and increase the denominator by one and write the resulting fraction. Compare each resulting fraction with the original. Explain why the resulting fractions are greater or less than the initial fraction.

Things to Think About

What happens to the size of the fraction when the numerator is increased? Increasing the numerator increases the number of parts being considered within the same whole amount and thus results in a larger fraction (e.g., $\frac{3}{4} > \frac{2}{4}$). Note that while the numerators differ, both fractions have a denominator of four. The "common denominator" technique that is used to compare fractions is based on the fact that if two fractions are divided into the same number of fractional pieces, then the larger fraction is the one with the most pieces.

What happens to the size of a fraction when the denominator is increased? Increasing the denominator increases the number of parts needed to make a whole. The fraction $\frac{2}{5}$ is smaller than $\frac{2}{4}$, since $\frac{2}{5}$ requires five parts to make a whole and $\frac{2}{4}$ requires only four parts to make a whole. Consider this set of fractions with increasing denominators: $\frac{2}{4}$, $\frac{2}{5}$, $\frac{2}{6}$, $\frac{2}{7}$, $\frac{2}{8}$, and $\frac{2}{9}$. Each fraction has a numerator of two, which indicates that there are two parts of some whole amount. However, as the denominator increases, the size of each of the two parts decreases. That is, $\frac{2}{5}$ is greater than $\frac{2}{9}$, because the parts or pieces in $\frac{2}{5}$ are larger than the parts or pieces in $\frac{2}{9}$. When two fractions have the same numerator (e.g., $\frac{3}{5}$ and $\frac{3}{7}$), you can compare them by simply considering the size of the parts; you do not need to find a common denominator! The "common numerator" technique is an efficient, conceptually based method for comparing fractions with identical numerators. For example, which of these fractions is greater: $\frac{8}{10}$ or $\frac{8}{12}$? In both fractions there are eight fractional pieces, but twelfths are smaller than tenths so $\frac{8}{10} > \frac{8}{12}$. ▲